

GENERALIZATIONS OF SOME INEQUALITIES FOR THE p -GAMMA, q -GAMMA AND k -GAMMA FUNCTIONS

KWARA NANTOMAH, EDWARD PREMPEH

ABSTRACT. In this paper, we present and prove some generalizations of some inequalities for the p -Gamma, q -Gamma and k -Gamma functions. Our approach makes use of the series representations of the psi, p -psi, q -psi and k -psi functions.

1. INTRODUCTION

We begin by recalling some basic definitions related to the Gamma function.

The Psi function, $\psi(t)$ is defined as,

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0. \quad (1)$$

where $\Gamma(t)$ is the classical Euler's Gamma function defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx, \quad t > 0. \quad (2)$$

The p -psi function, $\psi_p(t)$ is defined as,

$$\psi_p(t) = \frac{d}{dt} \ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0. \quad (3)$$

where $\Gamma_p(t)$ is the p -Gamma function defined by (see [3], [2])

$$\Gamma_p(t) = \frac{p! p^t}{t(t+1) \dots (t+p)} = \frac{p^t}{t(1 + \frac{t}{1}) \dots (1 + \frac{t}{p})}, \quad p \in N, \quad t > 0. \quad (4)$$

The q -psi function, $\psi_q(t)$ is defined as,

$$\psi_q(t) = \frac{d}{dt} \ln(\Gamma_q(t)) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \quad t > 0. \quad (5)$$

where $\Gamma_q(t)$ is the q -Gamma function defined by (see [5])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0, 1), \quad t > 0. \quad (6)$$

Similarly, the k -psi function, $\psi_k(t)$ is defined as follows.

$$\psi_k(t) = \frac{d}{dt} \ln(\Gamma_k(t)) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0. \quad (7)$$

2010 *Mathematics Subject Classification.* 33B15, 26A48.

Key words and phrases. p -Gamma Function, q -Gamma Function, k -Gamma Function, Inequality.

where $\Gamma_k(t)$ is the k -Gamma function defined by (see [1], [6])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x}{k}} x^{t-1} dx, \quad k > 0, \quad t > 0. \quad (8)$$

In [4], Krasniqi and Shabani proved the following results.

$$\frac{p^{-t} e^{-\gamma t} \Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_p(\alpha + t)} < \frac{p^{1-t} e^{\gamma(1-t)} \Gamma(\alpha + 1)}{\Gamma_p(\alpha + 1)} \quad (9)$$

for $t \in (0, 1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in [2], Krasniqi, Mansour and Shabani proved the following.

$$\frac{(1-q)^t e^{-\gamma t} \Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_q(\alpha + t)} < \frac{(1-q)^{t-1} e^{\gamma(1-t)} \Gamma(\alpha + 1)}{\Gamma_q(\alpha + 1)} \quad (10)$$

for $t \in (0, 1)$, where α is a positive real number such that $\alpha + t > 1$ and $q \in (0, 1)$.

In a recent paper [7], K. Nantomah also proved the following result.

$$\frac{k^{-\frac{t}{k}} e^{-t(\frac{k\gamma-\gamma}{k})} \Gamma(\alpha)}{\Gamma_k(\alpha)} \leq \frac{\Gamma(\alpha + t)}{\Gamma_k(\alpha + t)} \leq \frac{k^{\frac{1-t}{k}} e^{(1-t)(\frac{k\gamma-\gamma}{k})} \Gamma(\alpha + 1)}{\Gamma_k(\alpha + 1)} \quad (11)$$

for $t \in (0, 1)$, where α is a positive real number.

The main objective of this paper, is to establish and prove some generalizations of the inequalities (9), (10) and (11) as previously established in [4], [2] and [7] respectively.

2. PRELIMINARIES

We present the following auxiliary results.

Lemma 2.1. *The function $\psi(t)$ as defined in (1) has the following series representation.*

$$\psi(t) = -\gamma + (t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} = -\gamma - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{n(n+t)} \quad (12)$$

where γ is the Euler-Mascheroni's constant.

Proof. See [8].

Lemma 2.2. *The function $\psi_p(t)$ as defined in (3) has the following series representation.*

$$\psi_p(t) = \ln p - \sum_{n=0}^p \frac{1}{n+t} \quad (13)$$

Proof. See [4].

Lemma 2.3. *The function $\psi_q(t)$ as defined in (5) has the following series representation.*

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} \quad (14)$$

Proof. See [2].

Lemma 2.4. *The function $\psi_k(t)$ as defined in (7) also has the following series representation.*

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \quad (15)$$

Proof. See [7]

3. MAIN RESULTS

We now state and prove the results of this paper.

Lemma 3.1. *Let $a > 0$, $b > 0$ and $t > 1$. Then,*

$$a\gamma + b \ln p + a\psi(t) - b\psi_p(t) > 0$$

Proof. Using the series representations in equations (12) and (13) we have,

$$a\gamma + b \ln p + a\psi(t) - b\psi_p(t) = a(t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} + b \sum_{n=0}^p \frac{1}{(n+t)} > 0$$

Lemma 3.2. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma + b \ln p + a\psi(\alpha + \beta t) - b\psi_p(\alpha + \beta t) > 0$$

Proof. Follows directly from Lemma 3.1

Lemma 3.3. *Let $a > 0$, $b > 0$, $q \in (0, 1)$ and $t > 1$. Then,*

$$a\gamma - b \ln(1-q) + a\psi(t) - b\psi_q(t) > 0$$

Proof. Using the series representations in equations (12) and (14) we have,

$$a\gamma - b \ln(1-q) + a\psi(t) - b\psi_q(t) = a(t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} - b \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} > 0$$

Lemma 3.4. *Let $a > 0$, $b > 0$, $q \in (0, 1)$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma - b \ln(1-q) + a\psi(\alpha + \beta t) - b\psi_q(\alpha + \beta t) > 0$$

Proof. Follows directly from Lemma 3.3

Lemma 3.5. *Let $a \geq b > 0$, $k \geq 1$ and $t > 0$. Then,*

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k} \ln k + \frac{a-b}{t} + a\psi(t) - b\psi_k(t) \geq 0$$

Proof. Using the series representations in equations (12) and (15) we have,

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k} \ln k + \frac{a-b}{t} + a\psi(t) - b\psi_k(t) = t \left[a \sum_{n=1}^{\infty} \frac{1}{n(n+t)} - b \sum_{n=1}^{\infty} \frac{1}{nk(nk+t)} \right] \geq 0$$

Lemma 3.6. *Let $a \geq b > 0$, $k \geq 1$ and $\alpha + \beta t > 0$. Then,*

$$\frac{ka\gamma - b\gamma}{k} + \frac{b}{k} \ln k + \frac{a-b}{\alpha + \beta t} + a\psi(\alpha + \beta t) - b\psi_k(\alpha + \beta t) \geq 0$$

Proof. Follows directly from Lemma 3.5

Theorem 3.7. Define a function Ω by

$$\Omega(t) = \frac{p^{b\beta t} e^{a\beta\gamma t} \Gamma(\alpha + \beta t)^a}{\Gamma_p(\alpha + \beta t)^b}, \quad t \in (0, \infty), \quad p \in N \quad (16)$$

where a, b, α, β are positive real numbers such that $\alpha + \beta t > 1$. Then Ω is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$, the following inequalities are valid.

$$\frac{p^{-b\beta t} e^{-a\beta\gamma t} \Gamma(\alpha)^a}{\Gamma_p(\alpha)^b} < \frac{\Gamma(\alpha + \beta t)^a}{\Gamma_p(\alpha + \beta t)^b} < \frac{p^{b\beta(1-t)} e^{a\beta\gamma(1-t)} \Gamma(\alpha + \beta)^a}{\Gamma_p(\alpha + \beta)^b}. \quad (17)$$

Proof. Let $f(t) = \ln \Omega(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned} f(t) &= \ln \frac{p^{b\beta t} e^{a\beta\gamma t} \Gamma(\alpha + \beta t)^a}{\Gamma_p(\alpha + \beta t)^b} \\ &= b\beta t \ln p + a\beta\gamma t + a \ln \Gamma(\alpha + \beta t) - b \ln \Gamma_p(\alpha + \beta t) \end{aligned}$$

Then,

$$\begin{aligned} f'(t) &= a\beta\gamma + b\beta \ln p + a\beta\psi(\alpha + \beta t) - b\beta\psi_p(\alpha + \beta t) \\ &= \beta [a\gamma + b \ln p + a\psi(\alpha + \beta t) - b\psi_p(\alpha + \beta t)] > 0. \quad (\text{by Lemma 3.2}) \end{aligned}$$

That implies f is increasing on $t \in (0, \infty)$. Hence Ω is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\Omega(0) < \Omega(t) < \Omega(1)$$

yielding the result.

Theorem 3.8. Define a function ϕ by

$$\phi(t) = \frac{(1-q)^{-b\beta t} e^{a\beta\gamma t} \Gamma(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b}, \quad t \in (0, \infty), \quad q \in (0, 1) \quad (18)$$

where a, b, α, β are positive real numbers such that $\alpha + \beta t > 1$. Then ϕ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$, the following inequalities are valid.

$$\frac{(1-q)^{b\beta t} e^{-a\beta\gamma t} \Gamma(\alpha)^a}{\Gamma_q(\alpha)^b} < \frac{\Gamma(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b} < \frac{(1-q)^{b\beta(1-t)} e^{a\beta\gamma(1-t)} \Gamma(\alpha + \beta)^a}{\Gamma_q(\alpha + \beta)^b}. \quad (19)$$

Proof. Let $g(t) = \ln \phi(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned} g(t) &= \ln \frac{(1-q)^{-b\beta t} e^{a\beta\gamma t} \Gamma(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b} \\ &= -b\beta t \ln(1-q) + a\beta\gamma t + a \ln \Gamma(\alpha + \beta t) - b \ln \Gamma_q(\alpha + \beta t) \end{aligned}$$

Then,

$$\begin{aligned} g'(t) &= -b\beta \ln(1-q) + a\beta\gamma + a\beta\psi(\alpha + \beta t) - b\beta\psi_q(\alpha + \beta t) \\ &= \beta [a\gamma - b \ln(1-q) + a\psi(\alpha + \beta t) - b\psi_q(\alpha + \beta t)] > 0. \quad (\text{by Lemma 3.4}) \end{aligned}$$

That implies g is increasing on $t \in (0, \infty)$. Hence ϕ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\phi(0) < \phi(t) < \phi(1)$$

yielding the result.

Theorem 3.9. Define a function θ by

$$\theta(t) = \frac{(\alpha + \beta t)^{(a-b)} e^{t(\frac{ka\beta\gamma - b\beta\gamma}{k})} \Gamma(\alpha + \beta t)^a}{k^{-\frac{b\beta t}{k}} \Gamma_k(\alpha + \beta t)^b}, \quad t \in (0, \infty), \quad k \geq 1 \quad (20)$$

where a, b, α, β are positive real numbers such that $a \geq b$. Then θ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$, the following inequalities are valid.

$$\frac{\alpha^{(a-b)} e^{-t(\frac{ka\beta\gamma - b\beta\gamma}{k})} \Gamma(\alpha)^a}{(\alpha + \beta t)^{(a-b)} k^{\frac{b\beta t}{k}} \Gamma_k(\alpha)^b} \leq \frac{\Gamma(\alpha + \beta t)^a}{\Gamma_k(\alpha + \beta t)^b} \leq \frac{(\alpha + \beta)^{(a-b)} e^{(1-t)(\frac{ka\beta\gamma - b\beta\gamma}{k})} \Gamma(\alpha + \beta)^a}{(\alpha + \beta t)^{(a-b)} k^{\frac{b\beta}{k}(t-1)} \Gamma_k(\alpha + \beta)^b}. \quad (21)$$

Proof. Let $h(t) = \ln \theta(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned} h(t) &= \ln \frac{(\alpha + \beta t)^{(a-b)} e^{t(\frac{ka\beta\gamma - b\beta\gamma}{k})} \Gamma(\alpha + \beta t)^a}{k^{-\frac{b\beta t}{k}} \Gamma_k(\alpha + \beta t)^b} \\ &= (a-b) \ln(\alpha + \beta t) + \frac{b\beta t}{k} \ln k + t \left(\frac{ka\beta\gamma - b\beta\gamma}{k} \right) \\ &\quad + a \ln \Gamma(\alpha + \beta t) - b \ln \Gamma_k(\alpha + \beta t) \end{aligned}$$

Then,

$$\begin{aligned} h'(t) &= \frac{ka\beta\gamma - b\beta\gamma}{k} + \frac{b\beta}{k} \ln k + \beta \frac{a-b}{\alpha + \beta t} + a\beta\psi(\alpha + \beta t) - b\beta\psi_k(\alpha + \beta t) \\ &= \beta \left[\frac{ka\gamma - b\gamma}{k} + \frac{b}{k} \ln k + \frac{a-b}{\alpha + \beta t} + a\psi(\alpha + \beta t) - b\psi_k(\alpha + \beta t) \right] \geq 0 \end{aligned}$$

That is as a result of Lemma 3.6. That implies h is increasing on $t \in (0, \infty)$. Hence θ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\theta(0) \leq \theta(t) \leq \theta(1)$$

yielding the result.

4. CONCLUDING REMARKS

We dedicate this section to some remarks concerning our results.

Remark 4.1. If in Theorem 3.7 we set $a = b = \beta = 1$, then the inequalities in (9) are restored.

Remark 4.2. If in Theorem 3.8 we set $a = b = \beta = 1$, then the inequalities in (10) are restored.

Remark 4.3. If in Theorem 3.9 we set $a = b = \beta = 1$, then the inequalities in (11) are restored.

With the foregoing Remarks, the results of [4], [2] and [7] have been generalized.

REFERENCES

- [1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pachhammer k -symbol*, Divulgaciones Matemáticas **15**(2)(2007), 179-192.
- [2] V. Krasniqi, T. Mansour and A. Sh. Shabani, *Some Monotonicity Properties and Inequalities for Γ and ζ Functions*, Mathematical Communications **15**(2)(2010), 365-376.
- [3] V. Krasniqi and F. Merovci, *Logarithmically completely monotonic functions involving the generalized gamma function*, Le Matematiche **LXV**(2010), 15-23.
- [4] V. Krasniqi, A. Sh. Shabani, *Convexity Properties and Inequalities for a Generalized Gamma Function*, Applied Mathematics E-Notes **10**(2010), 27-35.
- [5] T. Mansour, *Some inequalities for the q -Gamma Function*, J. Ineq. Pure Appl. Math. **9**(1)(2008), Art. 18.
- [6] F. Merovci, *Power Product Inequalities for the Γ_k Function*, Int. Journal of Math. Analysis **4**(21)(2010), 1007-1012.
- [7] K. Nantomah, *Inequalities for the Psi and k -Gamma functions*, Global Journal of Mathematical Analysis, **2**(4)(2014), 290-296.
- [8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Camb. Univ. Press, 1969.

KWARA NANTOMAH

DEPARTMENT OF MATHEMATICS, UNIVERSITY FOR DEVELOPMENT STUDIES, NAVRONGO CAMPUS,
P. O. BOX 24, NAVRONGO, UE/R, GHANA.

E-mail address: mykwarasoft@yahoo.com, knantomah@uds.edu.gh

EDWARD PREMPEH

DEPARTMENT OF MATHEMATICS, KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY,
KUMASI, GHANA.

E-mail address: eprempeh.cos@knust.edu.gh